

No finite axiomatizations for posets embeddable into distributive lattices

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Abstract

Let m and n be cardinals with $3 \leq m, n \leq \omega$. We show that the class of posets that can be embedded into a distributive lattice via a map preserving all existing meets and joins with cardinalities strictly less than m and n respectively cannot be finitely axiomatized.

1 Introduction

Let m and n be cardinals with $3 \leq m, n \leq \omega$. It is shown in [8] that the problem of deciding whether a given finite poset can be embedded into a distributive lattice via a map preserving existing meets and joins with cardinalities strictly less than m and n respectively is NP -complete for all m and n except, possibly, the case where both m and n are equal to 3. Since polynomial time algorithms exist for checking whether a fixed first order sentence holds in a finite model (see [9] for proof of a more general result), if a class of posets with this kind of embedding property for some suitable m and n were finitely axiomatizable it would imply that $P = NP$. Needless to say, this implication strongly suggests that none of these classes is finitely axiomatizable. However, intuitive finite first order axiomatizations do exist for semilattices in similar situations [1, 6].

The equivalent (assuming finiteness or a suitable choice principle) question of embedding posets into rings of sets via maps preserving meets and joins smaller than specified (possibly infinite) cardinalities has been studied in [2, 3] using the terminology (m, n) -representable. In particular it was shown that all the classes where $m, n \leq \omega$ are elementary (though explicit axioms are not found) [3, theorem 4.5], but that in the cases where either m or n is equal to ω the corresponding class is not finitely axiomatizable (this also follows from the corresponding result for semilattices in [4]). However the cases where m and n are both finite remained open.

Since for $3 \leq m, n \leq \omega$ the classes of (m, n) -representable posets are all elementary, the question of whether they are finitely axiomatizable is equivalent to the question of whether their complement classes are elementary. By the Keisler-Shelah theorem [5, 7] the complement classes will be elementary if and only if they are closed under ultraproducts (as they must be closed under ultraroots and isomorphism).

For a poset P and cardinals α and β , the existence of an (α, β) -representation for P is equivalent to a separation property generalizing the separation of distributive lattices by prime filters (the Prime Ideal Theorem for distributive lattices). In this note we use this property to construct a sequence of finite posets, all of which fail to be $(3, 3)$ -representable, and an ultraproduct of this sequence which is (ω, ω) -representable, thus proving that the class of (m, n) -representable posets cannot be finitely axiomatizable for any choice of $n, m \geq 3$. The classes of (α, β) -representable posets when α and/or β are uncountable, and the classes where *all* meets and/or joins must be preserved, are known to not be elementary at all (though in some cases they are pseudoelementary) [2, 3].

In section 2 we introduce the basic notation, definitions and results for representable posets (using the notation of [3]). Finally in section 3 we construct the required sequence of posets and prove the necessary results to support our main claim.

2 Representable posets

We begin with some notational conventions. Given a poset P and a subset $S \subseteq P$ we define $S^\uparrow = \{p \in P : p \geq q \text{ for some } q \in S\}$. Given $p \in P$ we define $p^\uparrow = \{p\}^\uparrow$. Given a set I , an ultrafilter U of $\wp(I)$, and posets P_i for $i \in I$ we let $\prod_U P_i$ be the ultraproduct with respect to U . For an element of $\prod_U P_i$ we write, e.g. $[x] \in \prod_U P_i$.

Definition 2.1 ((α, β) -representable). *Let α and β be cardinals. We say a poset P is (α, β) -representable if there is a field of sets F and a 1-1 map $h : P \rightarrow F$ such that whenever S and T are subsets of P with $|S| < \alpha$ and $|T| < \beta$, if $\bigwedge S$ exists in P then $h(\bigwedge S) = \bigcap h[S]$, and if $\bigvee T$ exists in P then $h(\bigvee T) = \bigcup h[T]$. If $\alpha = \beta$ we just write α -representable.*

Definition 2.2 ((α, β) -filter). *Let α and β be cardinals, let P be a poset, and let γ be a subset of P . We say γ is an (α, β) -filter if whenever $S \subseteq \gamma$ and $|S| < \alpha$, if $\bigwedge S$ exists then $\bigwedge S \in \gamma$, and whenever $T \subseteq P$ with $|T| < \beta$, if $\bigvee T$ exists and $\bigvee T \in \gamma$ then $T \cap \gamma \neq \emptyset$. If $\alpha = \beta$ we just write α -filter.*

The following result relates (α, β) -representability to separation by (α, β) -filters. It appears explicitly in this form as [3, theorem 2.7], but the idea of using this kind of separation property for representability-like results for ordered structures has been in the literature for at least 50 years.

Theorem 2.3. *Let α and β be cardinals, and let P be a poset. Then P is (α, β) -representable if and only if for all $p, q \in P$, if $p \not\leq q$ then there is an (α, β) -filter $\gamma \subset P$ with $p \in \gamma$ and $q \notin \gamma$.*

Of course there is a dual result stated in terms of ideals rather than filters, and the details of this can also be found in [3].

The next lemma shows how we can translate the existence of certain (m, n) -filters into the existence of certain (m, n) -filters in an ultraproduct. It will play an important role in proving (m, n) -representability for our ultraproduct.

Lemma 2.4. *Let I be a set, and let U be a non-principal ultrafilter of $\wp(I)$. Let $3 \leq m, n \leq \omega$. For each $i \in I$ let P_i be a poset, and let $[a], [b] \in \prod_U P_i$. Let $u \in U$ and suppose that for all $i \in u$ there is an (m, n) -filter, γ_i , of P_i with $a(i) \in \gamma_i$ and $b(i) \notin \gamma_i$. Then there is an (m, n) -filter, Γ , of $\prod_U P_i$ with $[a] \in \Gamma$ and $[b] \notin \Gamma$.*

Proof. Let \mathcal{L} be the standard language of posets extended by the single unary predicate symbol G . In every poset P_i with $i \in u$ we interpret this predicate using

$$P_i \models G(p) \iff p \in \gamma_i$$

Then by definition we have $\prod_U P_i \models G([a])$ and $\prod_U P_i \not\models G([b])$. For every $i \in u$ we have $\{p \in P_i : G(p)\} = \gamma_i$, and is thus an (m, n) -filter. So P_i satisfies the set of first order sentences ensuring $\{p \in P_i : G(p)\}$ is an (m, n) -filter for all $i \in u$. Thus by Łoś' theorem $\{[c] \in \prod_U P_i : G([c])\}$ is also an (m, n) -filter, and since $G([a])$ but not $G([b])$ we are done. \square

3 Non-finite axiomatizability

We construct a sequence $(P_k : k = 0, 1, 2, \dots)$ of finite posets. Each of these posets fails to be 3-representable (and thus fails to be (m, n) -representable for all $m, n \geq 3$), but as k increases the posets become, in a sense, closer to being 3-representable. We then show that an ultraproduct of these posets is ω -representable (and so is (m, n) -representable for all $3 \leq m, n \leq \omega$). This shows that, for all $3 \leq m, n \leq \omega$, the complement of the class of (m, n) -representable posets is not elementary, and thus that the class of (m, n) -representable posets cannot be finitely axiomatized.

In order to construct P_k we first recursively define the sets N_n for $n \in \omega$ by

- $N_0 = \{a, b, c, d\}$
- Given N_n we define $N_{n+1} = \{n_{xy} : x \text{ and } y \text{ are distinct elements of } N_n\}$.
I.e. we get an element of N_{n+1} for every distinct pair of elements in N_n .

Then for all $n \in \omega$ we define

$$\hat{N}_n = \bigcup_{x \in N_n} \{x', x''\}$$

Given $k < \omega$ we define the carrier of P_k to be

$$\{p, q\} \cup \bigcup_{n=0}^k N_n \cup \bigcup_{n=0}^k \hat{N}_n$$

We assume of course that elements labeled differently are distinct. We define the order on P_k as follows:

1. $x < p$ for all $x \in N_0 = \{a, b, c, d\}$.

2. $x < x'$ and $x < x''$ for all $x \in N_n$ and for all $n \leq k$.
3. For all $1 \leq n \leq k$, if $x, y \in N_{n-1}$ and n_{xy} is the corresponding element of N_n we have $n_{xy} < x'$, $n_{xy} < x''$, $n_{xy} < y'$, and $n_{xy} < y''$.
4. For all $x \in N_k$ we have $q < x'$ and $q < x''$.
5. $x \leq x$ for all $x \in P_k$.

We now prove some facts about P_k , from which we deduce that P_k is indeed a poset, and also that it has certain features that will be useful to us.

Lemma 3.1. *If $x, y, z \in P_k$ and $x \leq y$ and $y \leq z$, then either $x = y$ or $y = z$.*

Proof. If $x \leq y$ then either (1) $x \in N_0$ and $y = p$, (2) $x \in N_n$ and $y \in \hat{N}_n$, (3) $x \in N_n$ and $y \in \hat{N}_{n-1}$, (4) $x = q$ and $y \in \hat{N}_k$, or (5) $x = y$. We note that p has no upper bound other than itself, and that this is also true for elements of \hat{N}_n for all $0 \leq n \leq k$. \square

Corollary 3.2. *P_k is a poset for all $k \in \omega$.*

Proof. Since reflexivity is automatic it remains only to check antisymmetry and transitivity, and these follow from lemma 3.1 \square

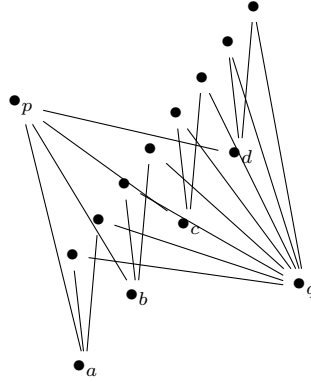


Figure 1: The poset P_0

Corollary 3.3. *The local maxima of P_k are precisely the elements of $\{p\} \cup \bigcup_{n=0}^k \hat{N}_n$, and the local minima of P_k are precisely the elements of $\{q\} \cup \bigcup_{n=0}^k N_k$.*

Proof. This is a restatement of the key observation in the proof of lemma 3.1. \square

Corollary 3.4. *The height of P_k is 2.*

Proof. This is equivalent to lemma 3.1. \square

Lemma 3.5. *Other than the joins of elements in $N_0 = \{a, b, c, d\}$ there are no non-trivial joins defined in P_k .*

Proof. Let $y \neq p \in P_k$ and let $X \subset P_k$ be such that $x < y$ for all $x \in X$. Then we must have $y \in \hat{N}_n$ for some $0 \leq n \leq k$, but then we cannot have $y = \bigvee X$ because by construction y has an incomparable twin with the same lower bounds. \square

Corollary 3.6. *Every element of $P_k \setminus (\{p\} \cup N_0)$ is join-prime.*

Figure 1 shows the poset P_0 . We obtain P_{k+1} from P_k by adding an extra row, $N_{k+1} \cup \hat{N}_{k+1}$, between $N_k \cup \hat{N}_k$ and q . N_{k+1} contains $\binom{|N_k|}{2}$ elements so the structures grow rapidly.

Lemma 3.7. *If γ is a 3-filter of P_k and γ contains at least 3 elements of N_n (for $0 \leq n < k$), then γ contains at least 3 members of N_{n+1} .*

Proof. Suppose $\{x, y, z\} \subset \gamma \cap N_n$. Then $\{x', x'', y', y'', z', z''\} \subset \gamma$ by up-closure, and so $\{n_{xy}, n_{xz}, n_{yz}\} \subset \gamma$ by closure under binary meets. \square

Proposition 3.8. *Let $x, y \in P_k$ and suppose $x \not\leq y$. Then the following are equivalent:*

1. $x \in \{p\} \cup N_0$ and $y \in \{q\} \cup \hat{N}_k$
2. there is no 3-filter containing x but not y
3. there is no ω -filter containing x but not y

Proof. First we show 1. \implies 2. directly. If γ is a 3-filter containing any one of $N_0 = \{a, b, c, d\}$ then it must also contain p by up-closure. So, by the condition on γ relating to joins it must also contain (at least) three members of N_0 . So by lemma 3.7 it must contain at least 3 members of N_k , and thus by up-closure and closure under binary meets it must also contain q , and hence by up-closure also every element of \hat{N}_k .

That 2. \implies 3. is automatic so we show 3. \implies 1. by proving the contrapositive. If $x \notin (\{p\} \cup N_0)$ then x^\uparrow is an ω -filter containing x but not y as x is join-prime. If $x \in (\{p\} \cup N_0)$ but $y \notin \{q\} \cup \hat{N}_k$ we can construct an ω -filter γ containing x but not y by making a suitable choice for which element of N_0 is left out of γ . To see this note that if we choose $z \in N_0$ and let $X = \{p\} \cup N_0 \setminus \{z\}$ then there is a smallest ω -filter containing X , γ_X say, generated deterministically by alternating closing upwards and closing under meets. It follows from the reasoning of lemma 3.7 that γ_X will contain exactly 3 elements of N_n for all $0 \leq n \leq k$. The key observation then is that if $e = e_{uv} \in N_n$ for some $u, v \in N_{n-1}$ and either $u \notin \gamma_X$ or $v \notin \gamma_X$ then $e \notin \gamma_X$. \square

Corollary 3.9. *P_k is not 3-representable for all $k \in \omega$*

Consider now the ultraproduct $\prod_U P_k$ where U is some non-principal ultrafilter over ω . Note that for $k < k'$ there is a natural identification of the carrier of P_k with a subset of the carrier of $P_{k'}$ (strictly speaking the former is already a subset of the latter). This identification is an order embedding except that the order between q and \hat{N}_k is broken. Using this identification, given any $k \in \omega$, and any $z \in P_k$, there is an element $[\bar{z}]$ of $\prod_U P_k$ defined by $\bar{z}(i) = z$ for all $i \geq k$.

Lemma 3.10. *Let $x \in P_k \setminus \{q\}$, and let $[y] \in \prod_U P_k$. Then $[\bar{x}] < [y]$ if and only if one of the following is true:*

1. $x \in N_0$ and $[y] = [\bar{p}]$,
2. $x \in N_n$ for some n and $[y] \in \{[\bar{x}'], [\bar{x}'']\}$, or
3. $x = e_{uv} \in N_{n+1}$ for some $u, v \in N_n$, and $[y] \in \{[\bar{u}'], [\bar{u}''], [\bar{v}'], [\bar{v}'']\}$

Proof. The ‘if’ part is trivial, so we prove ‘only if’. Since $[\bar{x}] < [y]$ we must have $x \in N_n$ for some n by corollaries 3.3 and 3.4. If $x \in N_0$ then $\{i \in \omega : y(i) = p\} \in U$ and so $[y] = [\bar{p}]$. More generally, if $x \in N_n$ then it has a finite set of upper bounds, and thus by properties of ultrafilters we must have either 2. or 3. as required. \square

Proposition 3.11. $\prod_U P_k$ is ω -representable.

Proof. Let $[x], [y] \in \prod_U P_k$ and suppose $[x] \not\leq [y]$. Suppose first that $\{i \in \omega : x(i) \notin \{p\} \cup N_0 \text{ or } y(i) \notin \{q\} \cup \hat{N}_i\} \in U$. Then by proposition 3.8 and lemma 2.4 there is an ω -filter of $\prod_U P_k$ containing $[x]$ but not $[y]$.

Suppose instead that $\{i \in \omega : x(i) \in \{p\} \cup N_0 \text{ and } y(i) \in \{q\} \cup \hat{N}_i\} \in U$. We define $\Gamma \subset \prod_U P_k$ by $\Gamma = \bigcup_{k \in \omega} \{[\bar{z}] : z \in P_k \setminus \{q\}\}$. We claim that Γ is an ω -filter of $\prod_U P_k$. That Γ is up-closed and closed under existing finite meets follows from lemma 3.10, and that its complement is closed under existing finite joins follows from properties of ultrafilters and lemma 3.5.

Now, since $\{i \in \omega : x(i) \in \{p\} \cup N_0\} \in U$ we must have $[x] = [\bar{z}]$ for some $z \in \{p\} \cup N_0 \subset P_0$ by properties of ultrafilters, and so $[x] \in \Gamma$. Since $\{i \in \omega : y(i) \in \{q\} \cup \hat{N}_i\} \in U$ we must have $[y] \neq [\bar{z}]$ for all $z \in P_k \setminus \{q\}$ for all $k \in \omega$ so by theorem 2.3 we are done. \square

Theorem 3.12. *For all m, n with $3 \leq m, n \leq \omega$ the class of (m, n) -representable posets is not finitely axiomatizable.*

Proof. We have shown that the complement class is not closed under ultraproducts and thus cannot be elementary by Łoś’ theorem. Hence the class of (m, n) -representable posets cannot be finitely axiomatized. \square

We note that the faint possibility remains that (m, n) -representability is finitely axiomatizable over the class of *finite* posets, so the study of first order axioms for classes of (m, n) -representable posets remains somewhat relevant to the P/NP question.

References

- [1] Balbes, R.: A representation theory for prime and implicative semilattices. *Trans. Amer. Math. Soc.* **136**, 261–267 (1969)
- [2] Egrot, R.: Non-elementary classes of representable posets. submitted
- [3] Egrot, R.: Representable posets. *J. Appl. Log.* **16**, 60–71 (2016). DOI doi:10.1016/j.jal.2016.03.003
- [4] Kearnes, K.: The class of prime semilattices is not finitely axiomatizable. *Semigroup Forum* **55**, 133–134 (1997)
- [5] Keisler, H.: Ultraproducts and elementary models. *Indag. Math.* **23**, 477–495 (1961)
- [6] Schein, B.: On the definition of distributive semilattices. *Algebra Universalis* **2**, 1–2 (1972)
- [7] Shelah, S.: Every two elementarily equivalent models have isomorphic ultrapowers. *Israel J. Math.* **10**, 224–233 (1971)
- [8] Van Alten, C.: Embedding ordered sets into distributive lattices. *Order* (to appear). DOI 10.1007/s11083-015-9376-6
- [9] Vardi, M.: On the complexity of bounded-variable queries. In: *ACM Symp. on Principles of Database Systems*, pp. 266–276. ACM press (1995)